Volume of the quantum mechanical state space

2006 J. Phys. A: Math. Gen. 39 13641

(http://iopscience.iop.org/0305-4470/39/44/003)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **39** (2006) 13641–13657 doi:10.1088/0305-4470/39/44/003

Volume of the quantum mechanical state space

Attila Andai

Department for Mathematical Analysis, Budapest University of Technology and Economics, H-1521 Budapest XI. Sztoczek u. 2, Hungary

E-mail: andaia@math.bme.hu

Received 25 May 2006, in final form 18 September 2006
Published 17 October 2006
Online at stacks.iop.org/JPhysA/39/13641

Abstract
The volume of the quantum mechanical state space over n-dimensional real, complex and quaternionic Hilbert spaces with respect to the canonical Euclidean measure is computed, and explicit formulae are presented for the expected value of the determinant also in the general setting. The case when the state space is endowed with a monotone metric or a pull-back metric is also considered; we give formulae for the volume of the state space with respect to the given Riemannian metric. We present the volume of the space of qubits with respect to various monotone metrics. It turns out that the volume of the space of qubits can also be infinite. We characterize those monotone metrics which generate infinite volume.

PACS numbers: 02.40.−k, 03.65.−w
Mathematics Subject Classification: 53C20, 81Q99

Introduction

The classical Jeffreys' prior is the square root of the determinant of the classical Fisher information matrix, up to a normalization constant. Analogously, the quantum mechanical counterpart of the Jeffreys' prior is the square root of the determinant of the quantum Fisher-information matrix. In the quantum mechanical case, one can endow the state space with different Riemannian metrics. Some of them are the monotone metrics, which can be labelled by special operator monotone functions. For each metric one has a Jeffreys' prior, so it is not unique as in the classical case. Different kinds of generalizations of the classical Jeffreys' prior lead to a different quantum mechanical Jeffreys' prior. This prior was widely examined by Slater if the metric is the Bures metric [19–24]. In this paper we compute the volume of the state space with respect to the Lebesgue measure, we give general formulae for the volume for monotone and pull-back metrics too and we characterize those monotone metrics which generate infinite volume of the state of qubits.

In the first section, we fix the notations for further computations and we mention some elementary lemmas which will be used in the following. In the second section, we compute the volume of the quantum mechanical state space over n-dimensional real, complex and quaternionic Hilbert spaces with respect to the canonical Euclidean measure. Before these general computations, we compute the volume of the state space over the three- and four-dimensional real Hilbert spaces to give insight into the general computational method. We explicitly present the expected value of the determinant in the general setting. Zyczkowski and Sommers gave a formula for the volume of the complex state space when it is endowed Hilbert–Schmidt measure [28]. We show that their result, which is based on the theory of random matrices, is fully compatible with the presented one up to a normalization constant. In the third section, we consider the case when the state space is endowed with a monotone metric or a pull-back metric, and we give formulae to compute the volume of the state space with respect to the given Riemannian metric. Finally, in the fourth section we deal with the qubit case. We present the volume of this space with respect to various monotone metrics. It turns out that the volume of the space of qubits can also be infinite. We characterize those monotone metrics which generate infinite volume. The more technical proofs can be found in the appendix.

1. Basic lemmas and notations

The quantum mechanical state space consists of real, complex or quaternionic self-adjoint positive matrices with trace 1. The state D is called a faithful state if every eigenvalue of D is strictly positive, or equivalently $D > 0$. We consider only the set of faithful states with real, complex and quaternionic entries:

$$\mathcal{M}_n^{\mathbb{R}} = \{X \in M(n, \mathbb{R}) \mid X = X^*, X > 0, \operatorname{Tr} X = 1\}$$
$$\mathcal{M}_n^{\mathbb{C}} = \{X \in M(n, \mathbb{C}) \mid X = X^*, X > 0, \operatorname{Tr} X = 1\}$$
$$\mathcal{M}_n^{\mathbb{H}} = \{X \in M(n, \mathbb{H}) \mid X = X^*, X > 0, \operatorname{Tr} X = 1\}.$$

The dimension of these state spaces are

$$\dim \mathcal{M}_n^{\mathbb{R}} = \frac{(n-1)(n+2)}{2} \qquad \dim \mathcal{M}_n^{\mathbb{C}} = n^2 - 1 \qquad \dim \mathcal{M}_n^{\mathbb{H}} = 2n^2 - n - 1.$$

The following lemmas will be our main tools; we will use them without mentioning and will also introduce some notations which will be used in the following.

The first lemma is about some elementary properties of the gamma function Γ.

Lemma 1. *Consider the function Γ, which can be defined for $z \in \mathbb{R}^+$ as*

$$\Gamma(z) = \int_0^\infty t^{z-1} \,\mathrm{e}^{-t} \,\mathrm{d}t.$$

This function has the following properties for every natural number $n \neq 0$ and real argument $z \in \mathbb{R}^+$:

$$\Gamma(n) = (n-1)! \qquad \Gamma(1+z) = z\Gamma(z) \qquad \Gamma(1/2) = \sqrt{\pi}$$
$$\Gamma(n+1/2) = \frac{(2n-1)!!}{2^n}\sqrt{\pi} \qquad \Gamma(n/2) = \frac{(n-2)!!}{2^{\frac{n-1}{2}}}\sqrt{\pi}.$$

For an $n \times n$ matrix A we set A_i to be the left upper $i \times i$ submatrix of A, where $i = 1, \ldots, n$. The next two lemmas are elementary proposition in linear algebra.

Lemma 2. *The $n \times n$ self-adjoint matrix A is positive definite if and only if the inequality $\det(A_i) > 0$ holds for every $i = 1, \ldots, n$.*

Lemma 3. *Assume that A is an $n \times n$ matrix with entries x ($x \in \mathbb{R}$) and B is a diagonal matrix with the elements B_{jj} on the main diagonal, then*

$$\det(A+B) = \det(B) + x \sum_{i=1}^{n} \prod_{\substack{j=1 \\ j\neq i}}^{n} B_{jj}.$$

Lemma 4. *Assume that A is an $n \times n$ self-adjoint, positive definite matrix with entries $(a_{ij})_{i,j=1,\dots,n}$ and the vector x consists of the first $(n-1)$ elements of the last column, that is, $x = (a_{1,n}, \dots, a_{n-1,n})$. Then for the matrix $T = \det(A_{n-1})(A_{n-1})^{-1}$, we have*

$$\det(A) = a_{nn} \det(A_{n-1}) - \langle x, Tx \rangle.$$

Proof. For elementary matrix computation, one should expand $\det(A)$ by minors, with respect to the last row. □

Lemma 5. *For parameters $a, b \in \mathbb{R}^+$ and $t \in \mathbb{R}^+$ the integral equalities*

$$\int_0^t x^a (t-x)^b \, \mathrm{d}x = t^{1+a+b} \frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+2)}$$

$$G_{a,b} := \int_0^1 x^a (1-x^2)^b \, \mathrm{d}x = \frac{1}{2} \frac{\Gamma(b+1)\Gamma\left(\frac{a+1}{2}\right)}{\Gamma\left(\frac{a}{2}+b+\frac{3}{2}\right)}$$

hold.

Proof. These are consequences of the formula below for the beta integral

$$\int_0^1 x^p (1-x)^q \, \mathrm{d}x = \frac{\Gamma(p+1)\Gamma(q+1)}{\Gamma(p+q+2)}.$$

□

Lemma 6. *The surface F_{n-1} of a unit sphere in an n-dimensional space is*

$$F_{n-1} = \frac{n\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)}.$$

Proof. It follows from the well-known formula for the volume of the sphere in n-dimension with radius r:

$$V_n(r) = \frac{r^n \pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)},$$

since $F_{n-1} = \frac{\mathrm{d}V_n(r)}{\mathrm{d}r}\big|_{r=1}$. □

When we integrate on a subset of the Euclidean space, we always integrate with respect to the usual Lebesgue measure. The Lebesgue measure on $\mathbb{R}^n$ will be denoted by λ_n.

Lemma 7. *Consider the simplex*

$$\Delta_{n-1} = \left\{ (x_1, \dots, x_n) \in]0,1[^n \;\middle|\; \sum_{k=1}^{n} x_k = 1 \right\},$$

then

$$\int_{\Delta_{n-1}} \left(\prod_{i=1}^{n} x_i \right)^k \mathrm{d}\lambda_{n-1}(x) = \frac{\Gamma(k+1)^n}{\Gamma(n(k+1))}.$$

Proof. The integral can be computed as

$$\int_{\Delta_{n-1}} \left(\prod_{i=1}^{n} x_i\right)^k \mathrm{d}\lambda_{n-1}(x) = \int_0^1 \int_0^{1-a_1} \cdots \int_0^{1-\sum_{j=1}^{n-2} a_j} \left(\prod_{i=1}^{n-1} a_i^k\right) \times \left[\left(1 - \sum_{i=1}^{n-2} a_i\right) - a_{n-1}\right]^k \mathrm{d}a_{n-1} \ldots \mathrm{d}a_2\, \mathrm{d}a_1.$$

Integrating with respect to a_{n-1}, the integral is

$$\frac{\Gamma(k+1)\Gamma(k+1)}{\Gamma(2k+2)} a_1^k a_2^k \cdots a_{n-2}^k ((1 - a_1 - \cdots - a_{n-3}) - a_{n-2})^{2k+1}$$

and in general, the ith integral is

$$\frac{\Gamma(k+1)\Gamma(ik+i)}{\Gamma((i+1)(k+1))} a_1^k a_2^k \cdots a_{n-1-i}^k ((1 - a_1 - \cdots - a_{n-2-i}) - a_{n-1-i})^{(i+1)k+i}.$$

Thus, the result is

$$\frac{\Gamma(k+1)\Gamma(k+1)}{\Gamma(2k+2)} \frac{\Gamma(k+1)\Gamma(2k+2)}{\Gamma(3k+3)} \frac{\Gamma(k+1)\Gamma(3k+3)}{\Gamma(4k+4)} \times \cdots \times \frac{\Gamma(k+1)\Gamma((n-1)(k+1))}{\Gamma(n(k+1))} = \frac{\Gamma(k+1)^n}{\Gamma(n(k+1))}.$$

□

Lemma 8. *Assume that T is an $n \times n$ self-adjoint, positive definite matrix and $k, \rho \in \mathbb{R}^+$. Set*

$$\begin{aligned}
\mathrm{E}_n^{\mathbb{R}}(T,\rho) &= \{x \in \mathbb{R}^n \mid \langle x, Tx\rangle < \rho\}, & T_{ij} &\in \mathbb{R};\\
\mathrm{E}_n^{\mathbb{C}}(T,\rho) &= \{x \in \mathbb{C}^n \mid \langle x, Tx\rangle < \rho\}, & T_{ij} &\in \mathbb{C};\\
\mathrm{E}_n^{\mathbb{H}}(T,\rho) &= \{x \in \mathbb{H}^n \mid \langle x, Tx\rangle < \rho\}, & T_{ij} &\in \mathbb{H};
\end{aligned}$$

then

$$\int_{E_n^{\mathbb{R}}(T,\rho)} (\rho - \langle x, Tx\rangle)^k \,\mathrm{d}\lambda_n(x) = \frac{\rho^{\frac{n}{2}+k}}{\sqrt{\det(T)}} F_{n-1} G_{n-1,k},$$

$$\int_{E_{2n}^{\mathbb{C}}(T,\rho)} (\rho - \langle x, Tx\rangle)^k \,\mathrm{d}\lambda_{2n}(x) = \frac{\rho^{n+k}}{\det(T)} F_{2n-1} G_{2n-1,k},$$

$$\int_{E_{4n}^{\mathbb{H}}(T,\rho)} (\rho - \langle x, Tx\rangle)^k \,\mathrm{d}\lambda_{4n}(x) = \frac{\rho^{2n+k}}{\det(T)^2} F_{4n-1} G_{4n-1,k}.$$

Proof. We prove the statement for the real case only; the other cases can be proved in the same way. The set $\mathrm{E}_n^{\mathbb{R}}(T,\rho)$ is an n-dimensional ellipsoid, so to compute the integral first we transform our canonical basis to a new one, which is parallel to the axes of the ellipsoid. Since this is an orthogonal transformation, its Jacobian is 1. When we transform this ellipsoid to a unit sphere, the Jacobian of this transformation is

$$\prod_{k=1}^{n} \sqrt{\frac{\rho}{\mu_k}},$$

where $(\mu_k)_{k=1,\ldots,n}$ are the eigenvalues of T. Then we compute the integral in spherical coordinates. The integral with respect to the angles give the surface of the sphere $F_{n-1} r^{n-1}$, and the radial part is

$$\int_0^1 F_{n-1} r^{n-1} \sqrt{\frac{\rho^n}{\det(T)}} (\rho - \rho r^2)^k \,\mathrm{d}r.$$

□

2. Volume of the state space with respect to the Lebesgue measure

Before investigating the general setting, we compute the volume of the spaces $\mathcal{M}_3^{\mathbb{R}}$ and $\mathcal{M}_4^{\mathbb{R}}$. For a matrix with real entries

$$A = \begin{pmatrix} a & f & h \\ f & b & g \\ h & g & c \end{pmatrix}$$

we set $A_1 = (a)$, $A_2 = \left(\begin{smallmatrix} a & f \\ f & b \end{smallmatrix}\right)$, $A_3 = A$ and D denotes the matrix, which contains only the diagonal elements of A, that is $D_{ij} = \delta_{ij} A_{ii}$. The matrix A is in $\mathcal{M}_3^{\mathbb{R}}$ if and only if the following set of inequalities hold:

$$\begin{aligned}
&\det(A_1) = a > 0 \qquad a + b + c = 1 \\
&\det(A_2) = ab - f^2 > 0 \\
&\det(A_3) = abc + 2fgh - h^2 b - g^2 a - f^2 c > 0.
\end{aligned}$$

These inequalities can be rewritten as

$$(a, b, c) \in \Delta_2, \qquad \langle (f), T_1(f)\rangle < b \det(A_1), \qquad \langle (h, g), T_2(h, g)\rangle < c \det(A_2),$$

where $T_i = \det(A_i) A_i^{-1}$ for $i = 1, 2$. It means that for fixed D and A_2 the parameters (h, g) are in $\mathrm{E}_2^{\mathbb{R}}(T_2, c \det(A_2))$. We set $V(A_2)$ to be equal to the volume of the parameter space of (h, g) if D and A_2 are given, that is,

$$V(A_2) = \int_{\mathrm{E}_2^{\mathbb{R}}(T_2, c\det(A_2))} 1 \,\mathrm{d}g \,\mathrm{d}h = \frac{c \det(A_2)}{\sqrt{\det(T_2)}} \frac{\pi}{\Gamma(2)} = \pi c \sqrt{\det(A_2)}.$$

If D and A_1 are fixed, then we set $V(A_1)$ to be equal to the volume of the parameter space (f, g, h):

$$V(A_1) = \int_{\mathrm{E}_1^{\mathbb{R}}(T_1, b\det(A_1))} V(A_2) \,\mathrm{d}f = \int_{-\sqrt{ab}}^{\sqrt{ab}} \pi c \sqrt{ab - f^2} \,\mathrm{d}f = \frac{\pi^2}{2} abc.$$

Finally, the volume of the $\mathcal{M}_3^{\mathbb{R}}$ space is

$$V\big(\mathcal{M}_3^{\mathbb{R}}\big) = \int_{\Delta_2} V(A_1) \,\mathrm{d}\lambda_2 = \frac{\pi^2}{2} \int_0^1 \int_0^{1-a} ab(1 - a - b) \,\mathrm{d}b \,\mathrm{d}a = \frac{\pi^2}{240}.$$

A 4×4 real, symmetric matrix with diagonal elements a_1, a_2, a_3 and a_4 is an element of the space $\mathcal{M}_4^{\mathbb{R}}$ if and only if

$$\begin{aligned}
&\det(A_1) = a_1 > 0 \quad \sum_{k=1}^{4} a_k = 1 \\
&\det(A_2) = a_2 \det(A_1) - \langle x_1, T_1 x_1\rangle > 0 \\
&\det(A_3) = a_3 \det(A_2) - \langle x_2, T_2 x_2\rangle > 0 \\
&\det(A_4) = a_4 \det(A_3) - \langle x_3, T_3 x_3\rangle > 0.
\end{aligned}$$

We set $T_i = \det(A_i) A_i^{-1}$ for $i = 1, 2, 3$. For fixed parameters A_3 and D,

$$V(A_3) = \int_{\mathrm{E}_3^{\mathbb{R}}(T_3, a_4 \det(A_3))} 1 \,\mathrm{d}\lambda_3 = a_4^{3/2} F_2 G_{2,0} \sqrt{\det(A_3)},$$

where we used the notation of the previous example; in this case $V(A_3)$ is the volume of the space of those parameters which do not belong to A_3 and D. Now assume that A_2 and D are given, then

$$
\begin{aligned}
V(A_2) &= \int_{\mathrm{E}_2^{\mathbb{R}}(T_2, a_3 \det(A_2))} a_4^{3/2} F_2 G_{2,0} \sqrt{\det(A_3)}\, \mathrm{d}\lambda_2 \\
&= F_2 G_{2,0} a_4^{3/2} \int_{\mathrm{E}_2^{\mathbb{R}}(T_2, a_3 \det(A_2))} (a_3 \det(A_2) - \langle x, T_2 x\rangle)^{\frac{1}{2}}\, \mathrm{d}\lambda_2(x) \\
&= F_2 F_1 G_{2,0} G_{1,1/2} a_4^{3/2} a_3^{3/2} \det(A_2).
\end{aligned}
$$

If D is given

$$
\begin{aligned}
V(A_1) &= \int_{\mathrm{E}_1^{\mathbb{R}}(T_1, a_2 \det(A_1))} F_2 F_1 G_{2,0} G_{1,1/2} a_4^{3/2} a_3^{3/2} \det(A_2)\, \mathrm{d}\lambda_1 \\
&= F_2 F_1 G_{2,0} G_{1,1/2} a_4^{3/2} a_3^{3/2} \int_{\mathrm{E}_1^{\mathbb{R}}(T_1, a_2 \det(A_1))} (a_2 \det(A_1) - \langle x, T_1 x\rangle)\, \mathrm{d}\lambda_1(x) \\
&= F_2 F_1 F_0 G_{2,0} G_{1,1/2} G_{0,1} (a_1 a_2 a_3 a_4)^{3/2}.
\end{aligned}
$$

Since

$$
\int_{\Delta_3} (a_1 a_2 a_3 a_4)^{3/2}\, \mathrm{d}\lambda_3(a) = \frac{\Gamma(3/2+1)^4}{\Gamma(10)},
$$

the volume of the four-dimensional real state space is

$$
V\left(\mathcal{M}_4^{\mathbb{R}}\right) = F_2 F_1 F_0 G_{2,0} G_{1,1/2} G_{0,1} \frac{\Gamma(3/2+1)^4}{\Gamma(10)} = \frac{3\pi^4}{8 \cdot 9!}.
$$

The rather technical proofs of the following theorems can be found in the appendix. The idea behind the proofs can be understood by the above-mentioned examples.

Theorem 1. *For every $k \in \mathbb{N}$, the volumes of the state spaces $\mathcal{M}_{2k}^{\mathbb{R}}$ and $\mathcal{M}_{2k+1}^{\mathbb{R}}$ are*

$$
\begin{aligned}
V\left(\mathcal{M}_{2k}^{\mathbb{R}}\right) &= \frac{\pi^{k^2}}{2^{k^2+k}} \frac{(2k)!}{k!(2k^2+k-1)!} \prod_{i=1}^{k-1} (2i)! \\
V\left(\mathcal{M}_{2k+1}^{\mathbb{R}}\right) &= \left(\frac{\pi}{2}\right)^{k^2+k} \frac{(2k)!}{(k-1)!(2k^2+3k)!} \prod_{i=1}^{k-1} (2i)!.
\end{aligned}
$$

Theorem 2. *For every $n \in \mathbb{N}$, the volume of the state space $\mathcal{M}_n^{\mathbb{C}}$ is*

$$
V\left(\mathcal{M}_n^{\mathbb{C}}\right) = \frac{\pi^{\frac{n(n-1)}{2}}}{(n^2-1)!} \prod_{i=1}^{n-1} i!.
$$

Theorem 3. *For every $n \in \mathbb{N}$, the volume of the state space $\mathcal{M}_n^{\mathbb{H}}$ is*

$$
V\left(\mathcal{M}_n^{\mathbb{H}}\right) = \frac{(2n-2)!\pi^{n^2-n}}{(2n^2-n-1)!} \prod_{i=1}^{n-2} (2i)!.
$$

A slight modification of the proofs of the above-mentioned theorems gives the following theorem.

Theorem 4. *For every parameter* $\alpha \in \mathbb{R}^+$ *and* $n \in \mathbb{N}$*, the expected value of the function* $\det^\alpha$ *on the state spaces* $\mathcal{M}_n^{\mathbb{R}}$, $\mathcal{M}_n^{\mathbb{C}}$ *and* $\mathcal{M}_n^{\mathbb{H}}$ *with respect to the normalized Lebesgue measures* $\mu_{\mathbb{R}}$, $\mu_{\mathbb{C}}$ *and* $\mu_{\mathbb{H}}$ *are*

$$\int_{\mathcal{M}_n^{\mathbb{R}}} \det(A)^\alpha \, \mathrm{d}\mu_{\mathbb{R}}(A) = \frac{\Gamma\left(\frac{n^2+n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)} \frac{\Gamma\left(\frac{n+1}{2}+\alpha\right)}{\Gamma\left(\frac{n^2+n}{2}+n\alpha\right)} \prod_{i=1}^{n-1} \frac{\Gamma\left(\frac{i+1}{2}+\alpha\right)}{\Gamma\left(\frac{i+1}{2}\right)}$$

$$\int_{\mathcal{M}_n^{\mathbb{C}}} \det(A)^\alpha \, \mathrm{d}\mu_{\mathbb{C}}(A) = \frac{(n^2-1)!}{(n-1)!} \frac{\Gamma(n+\alpha)}{\Gamma(n^2+n\alpha)} \prod_{i=1}^{n-1} \frac{\Gamma(i+\alpha)}{\Gamma(i)}$$

$$\int_{\mathcal{M}_n^{\mathbb{H}}} \det(A)^\alpha \, \mathrm{d}\mu_{\mathbb{H}}(A) = \frac{\Gamma(2n^2-n)}{\Gamma(2n-1)} \frac{\Gamma(2n+\alpha-1)}{\Gamma(2n^2-n+\alpha n)} \prod_{i=1}^{n-1} \frac{\Gamma(2i-1+\alpha)}{\Gamma(2i-1)}.$$

The volume of $\mathcal{M}_n^{\mathbb{C}}$ was computed by Zyczkowski and Sommers with respect to the Hilbert–Schmidt measure [28]. They used some elements of the theory of random matrices, for example, Hall's joint probability distribution [10] and some integral formulae from the book of Mehta [13]. The parametrization of the state space $\mathcal{M}_n^{\mathbb{C}}$ in their approach is

$$D = \frac{1}{n} I + \sum_{k=1}^{n^2-1} \tau_k \lambda_k,$$

where $(\lambda_k)_{k=1,\ldots,n^2-1}$ are the traceless self-adjoint generators of $SU(n)$, which fulfill the normalization $\operatorname{Tr} \lambda_k \lambda_l = \delta_{kl}$, $(\tau_k)_{k=1,\ldots,n^2-1}$ are the real parameters, and I denotes the identity matrix. In this setting, the parametrization of the space $\mathcal{M}_2^{\mathbb{C}}$ is the following:

$$\begin{pmatrix} \frac{1}{2} + \frac{z}{\sqrt{2}} & \frac{x}{\sqrt{2}} + \mathrm{i}\frac{y}{\sqrt{2}} \\ \frac{x}{\sqrt{2}} - \mathrm{i}\frac{y}{\sqrt{2}} & \frac{1}{2} - \frac{z}{\sqrt{2}} \end{pmatrix}$$

A matrix given by the above formula is a state if and only if $x^2 + y^2 + z^2 \leqslant 1/\sqrt{2}$. In this setting, the volume of the space of complex qubits is

$$V^{(\mathrm{Z.S.})}\left(\mathcal{M}_2^{\mathbb{C}}\right) = \frac{\sqrt{2}\pi}{3}$$

and, in general, the volume of $\mathcal{M}_n^{\mathbb{C}}$ is given by equation (4.5) in [28]:

$$V^{(\mathrm{Z.S.})}\left(\mathcal{M}_n^{\mathbb{C}}\right) = \sqrt{n} 2^{\frac{n(n-1)}{2}} \frac{\pi^{\frac{n(n-1)}{2}}}{(n^2-1)!} \prod_{i=1}^{n-1} i!.$$

The difference between this and our result is given by the next equation.

$$\frac{V^{(\mathrm{Z.S.})}\left(\mathcal{M}_n^{\mathbb{C}}\right)}{V\left(\mathcal{M}_n^{\mathbb{C}}\right)} = \sqrt{n} 2^{\frac{n(n-1)}{2}}.$$

The difference in the normalization factor can be understood by the decomposition of the Hilbert–Schmidt measure according to equation (3.7) in [28]:

$$\mathrm{d}V_{\mathrm{HS}} = \mathrm{d}\mu(\Lambda_1, \Lambda_2, \ldots, \Lambda_n) \times \mathrm{d}\nu_{\mathrm{Haar}},$$

where the first factor depends on the eigenvalues $(\Lambda_i)_{i=1,\ldots,n}$ of the state and ν_{Haar} is a Haar measure on the unitary group. This parametrization causes a normalization factor

$$2^{\frac{n(n-1)}{2}}$$

in the Haar measure with respect to the Lebesgue measure according to equation (A3) in [28]. The Riemannian metric induced by the Hilbert–Schmidt measure on the $(n-1)$-dimensional simplex $(\Lambda_i)_{i=1,\dots,n}$ has determinant n according to the conclusion after equation (3.5) in [28]. Since the volume element gains with the square root of the determinant of the Riemannian metric, it gives the normalization factor $\sqrt{n}$. It means that our result is fully compatible with the computation given by Zyczkowski and Sommers.

3. Volume of the state space endowed with Riemannian metrics

To simplify the notations, the set of real or complex self-adjoint matrices will be denoted by M_n, the set of traceless real or complex self-adjoint matrices will be denoted by $M_n^{(0)}$ and the set of real or complex states by $\mathcal{M}_n$. The space M_n has a natural differentiable structure; the tangent space T_D at $D \in M_n$ can be identified with M_n. The space $\mathcal{M}_n$ can be endowed with a differentiable structure too [11] and the tangent space T_D at $D \in \mathcal{M}_n$ can be identified with $M_n^{(0)}$.

A map

$$g : \mathcal{M}_n \times M_n^{(0)} \times M_n^{(0)} \to \mathbb{C} \qquad (D, X, Y) \mapsto g_D(X, Y)$$

will be called a Riemannian metric if the following condition holds. For all $D \in \mathcal{M}_n$, the map

$$g_D : M_n^{(0)} \times M_n^{(0)} \to \mathbb{C} \qquad (X, Y) \mapsto g_D(X, Y)$$

is a scalar product and for all $X \in M_n^{(0)}$, the map

$$g_{\cdot}(X, X) : \mathcal{M}_n \to \mathbb{C} \qquad D \mapsto g_D(X, X)$$

is smooth. We now use differential geometrical notation to define the volume of the Riemannian manifold $(\mathcal{M}_n, g)$. In this case, the Riemannian metric is a

$$g : \mathcal{M}_n \to \mathrm{LIN}\left(M_n^{(0)} \times M_n^{(0)}, \mathbb{R}\right) \qquad D \mapsto ((X, Y) \mapsto K_D(X, Y))$$

map, where $\mathrm{LIN}(U, V)$ denotes the set of linear maps from the vector space U to the vector space V. From the metric g, we can construct the function

$$\det(g) : \mathcal{M}_n \to \mathbb{R}$$

which is strictly positive at every point, since for every $D \in \mathcal{M}_n$ the map $g(D)$ defines a scalar product on the vector space T_D. The volume of a Riemannian manifold $(\mathcal{M}_n, g)$ is defined as

$$\int_{\mathcal{M}_n} \sqrt{\det(g(D))}\, \mathrm{d}\lambda_{\dim(\mathcal{M}_n)}(D).$$

The volume of the manifold is invariant with respect to the parametrization.

Čencov and Morozova [6, 15] were the first to study the monotone metrics on classical statistical manifolds. They proved that such a metric is unique, up to normalization. The noncommutative extension of the Čencov theorem was given by Petz [16]. Stochastic maps are the counterpart of Markovian maps in this setting. A linear map between matrix spaces $T : M_n \to M_m$ is called a stochastic map if it is trace preserving and completely positive.

Theorem 5. *Consider the family of Riemannian manifolds $(\mathcal{M}_n, g_n)_{n\in\mathbb{N}}$. If for every stochastic map $T : M_n \to M_m$ the following monotonicity property holds*

$$g_{T(D)}(T(X), T(X)) \leqslant g_D(X, X) \qquad \forall D, X \in M_n,$$

then there exists an operator monotone function $f : \mathbb{R}^+ \to \mathbb{R}$ with the property $f(x) = xf(x^{-1})$, such that

$$g_D(X, Y) = \mathrm{Tr}\left(X\left(R_{n,D}^{\frac{1}{2}} f(L_{n,D} R_{n,D}^{-1}) R_{n,D}^{\frac{1}{2}}\right)^{-1}(Y)\right),$$

for all $n \in \mathbb{N}$ where $L_{n,D}(X) = DX$, $R_{n,D}(X) = XD$ for all $D, X \in M_n$.

These metrics are considered as the noncommutative generalizations of the Fisher information. These metrics are called monotone metrics. It means that there exists a bijective mapping between the monotone family of metrics and some operator monotone functions. We use the normalization condition $f(1) = 1$ for the function f in the previous theorem.

Let $D \in \mathcal{M}_n$ and choose a basis of $\mathbb{R}^n$ such that $D = \sum_{j=1}^{n} \mu_j E_{jj}$ is diagonal, where $(E_{jk})_{1\leqslant j,k\leqslant n}$ is the usual system of matrix units. Let us define the following self-adjoint matrices:

$$\begin{aligned} F_{jk} &= E_{jk} + E_{kj} \qquad & 1 \leqslant j \leqslant k \leqslant n \\ H_{jk} &= \mathrm{i}\, E_{jk} - \mathrm{i}\, E_{kj} \qquad & 1 \leqslant j < k \leqslant n. \end{aligned}$$

The set of matrices $(F_{ij})_{1\leqslant i\leqslant j\leqslant n} \cup (H_{ij})_{1\leqslant i<j\leqslant n}$ form a basis of the tangent space at D for complex matrices and $(F_{ij})_{1\leqslant i\leqslant j\leqslant n}$ form a basis for real ones. We have for the metric from [14] that

$$\begin{aligned} &\text{if} \quad 1 \leqslant i < j \leqslant n, 1 \leqslant k < l \leqslant n: && \begin{cases} g(D)(H_{ij}, H_{kl}) = \delta_{ik}\delta_{jl} 2m(\mu_i, \mu_j) \\ g(D)(F_{ij}, F_{kl}) = \delta_{ik}\delta_{jl} 2m(\mu_i, \mu_j) \\ g(D)(H_{ij}, F_{kl}) = 0, \end{cases} \\ &\text{if} \quad 1 \leqslant i < j \leqslant n, 1 \leqslant k \leqslant n: && g(D)(H_{ij}, F_{kk}) = g(D)(F_{ij}, F_{kk}) = 0, \\ &\text{if} \quad 1 \leqslant i \leqslant n, 1 \leqslant k \leqslant n: && g(D)(F_{ii}, F_{kk}) = \delta_{ik} 4m(\mu_i, \mu_i), \end{aligned}$$

where

$$m(\mu_i, \mu_j) = \frac{1}{\mu_j f\left(\frac{\mu_i}{\mu_j}\right)}.$$

We use the canonical parametrization for the off-diagonal elements of $\mathcal{M}_n$ and $(x_1, \ldots, x_{n-1}, 1 - (x_1 + \cdots + x_{n-1}))$ for the diagonal ones. The corresponding tangent vectors for the diagonal coordinates are $A_i = E_{ii} - E_{nn}$ for $i = 1, \ldots, n-1$. Since $g(D)(A_i, A_j) = g(D)(E_{ii}, E_{jj}) + g(D)(E_{nn}, E_{nn}) = \delta_{ij}\frac{1}{\mu_i} + \frac{1}{\mu_n}$, the determinant of the metric is

$$\begin{aligned} \det(g(D)) &= \left(\prod_{1\leqslant i<j\leqslant n} 2m_{ij} \right) \varphi \qquad \text{if} \quad D \in \mathcal{M}_n^{\mathbb{R}} \\ \det(g(D)) &= \left(\prod_{1\leqslant i<j\leqslant n} 4m_{ij}^2 \right) \varphi \qquad \text{if} \quad D \in \mathcal{M}_n^{\mathbb{C}}, \end{aligned}$$

where φ is a determinant of an $(n-1) \times (n-1)$ matrix:

$$\varphi = \det \begin{pmatrix} \frac{1}{\mu_1} + \frac{1}{\mu_n} & \frac{1}{\mu_n} & \cdots & \frac{1}{\mu_n} \\ \frac{1}{\mu_n} & \frac{1}{\mu_2} + \frac{1}{\mu_n} & \cdots & \frac{1}{\mu_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\mu_n} & \frac{1}{\mu_n} & \cdots & \frac{1}{\mu_{n-1}} + \frac{1}{\mu_n} \end{pmatrix} = \frac{1}{\det(D)}.$$

Theorem 6. *The volume of the real and complex state space endowed with a Riemannian metric which is generated by the operator monotone function f is*

$$V\left(\mathcal{M}_n^{\mathbb{R}}, g_f\right) = 2^{\frac{n(n-1)}{4}} \int_{\mathcal{M}_n^{\mathbb{R}}} \frac{1}{\sqrt{\det(D)}} \left(\prod_{1\leqslant i<j\leqslant n} m(\mu_i(D), \mu_j(D))^{\frac{1}{2}} \right) \mathrm{d}\lambda_{\dim(\mathcal{M}_n^{\mathbb{R}})}(D)$$

$$V\left(\mathcal{M}_n^{\mathbb{C}}, g_f\right) = 2^{\frac{n(n-1)}{2}} \int_{\mathcal{M}_n^{\mathbb{C}}} \frac{1}{\sqrt{\det(D)}} \left(\prod_{1\leqslant i<j\leqslant n} m(\mu_i(D), \mu_j(D)) \right) \mathrm{d}\lambda_{\dim(\mathcal{M}_n^{\mathbb{C}})}(D).$$

Consider the operator monotone function $f_1(x) = (1+x)/2$ which generates the Bures metric. In this case, the volume of the complex state space was computed by Sommers and Zyczkowski [26]. There is a conjecture about the volume of the complex state when it is endowed by the Kubo–Mori metric, which is generated by the function $f_2(x) = (x-1)/\log x$. Slater conjectured [25] on the basis of numerical evidence that

$$V\left(\mathcal{M}_n^{\mathbb{C}}, g_{f_2}\right) = 2^{\frac{n(n-1)}{2}} V\left(\mathcal{M}_n^{\mathbb{C}}, g_{f_1}\right).$$

We can endow the Riemannian space with a pull-back metric too. Consider the functions $h :]0,1[\to \mathbb{R}$ with analytic continuation on a neighbourhood of the $]0,1[$ interval and suppose that $h'(x) \neq 0$ for every $x \in]0,1[$. We call such functions admissible functions. The space M_n will geometrically be considered a Riemannian space $(\mathbb{R}^d, g_E)$, where $d_{\mathbb{R}} = \frac{(n-1)(n+2)}{2}$ for real matrices and $d_{\mathbb{C}} = n^2 - 1$ for complex ones and g_E is the canonical Riemannian metric on M_n. That is, at every point $D \in \mathcal{M}_n$ for every vectors $X, Y \in \mathcal{M}_n$ in the tangent space at D the metric is

$$g_E(D)(X, Y) = \operatorname{Tr} XY.$$

For an admissible function $h :]0,1[\to \mathbb{R}$, the pull-back geometry of the spaces $\mathcal{M}_n^{\mathbb{R}}$ and $\mathcal{M}_n^{\mathbb{C}}$ is the Riemannian geometry g_h induced by the map

$$\phi_{h,n} : \mathcal{M}_n \to M_n \qquad D \mapsto h(D).$$

This Riemannian space will be denoted by $(\mathcal{M}_n, g_h)$.

For example, if the functions h are $p\sqrt[p]{x}$ if $p \neq 0$ or $\log x$, then we get the α-geometries [7, 9].

If $D \in \mathcal{M}_n$ is diagonal, i.e. $D = \sum_{i=1}^n \mu_i E_{ii}$, then the metric can be computed (see [4]) as

$$\begin{array}{ll}
\text{if} \quad 1 \leqslant i < j \leqslant n, 1 \leqslant k < l \leqslant n : & \begin{cases} g(D)(H_{ij}, H_{kl}) = \delta_{ik}\delta_{jl} 2M(\mu_i, \mu_j)^2 \\ g(D)(F_{ij}, F_{kl}) = \delta_{ik}\delta_{jl} 2M(\mu_i, \mu_j)^2 \\ g(D)(H_{ij}, F_{kl}) = 0, \end{cases} \\
\text{if} \quad 1 \leqslant i < j \leqslant n, 1 \leqslant k \leqslant n : & g(D)(H_{ij}, F_{kk}) = g(D)(F_{ij}, F_{kk}) = 0, \\
\text{if} \quad 1 \leqslant i \leqslant n, 1 \leqslant k \leqslant n : & g(D)(F_{ii}, F_{kk}) = \delta_{ik} 4M(\mu_i, \mu_i)^2,
\end{array}$$

where

$$M(\mu_i, \mu_j) = \begin{cases} \dfrac{h(\mu_i) - h(\mu_j)}{\mu_i - \mu_j} & \text{if} \quad \mu_i \neq \mu_j \\ h'(\mu_i) & \text{if} \quad \mu_i = \mu_j. \end{cases}$$

Using the previous considerations about the volume of the state space, we have the following theorem.

Theorem 7. *The volume of the real and complex state space endowed with a pull-back metric* g_h *is*

$$V(\mathcal{M}_n^{\mathbb{R}}) = 2^{\frac{n(n-1)}{4}} \int_{\mathcal{M}_n^{\mathbb{R}}} \sqrt{\sum_{i=1}^{n} \prod_{\substack{j=1\\ j\neq i}}^{n} h\prime(\mu_j)} \left(\prod_{1\leqslant i<j\leqslant n} M(\mu_i(D), \mu_j(D)) \right) \mathrm{d}\lambda_{\dim(\mathcal{M}_n^{\mathbb{R}})}(D)$$

$$V(\mathcal{M}_n^{\mathbb{C}}) = 2^{\frac{n(n-1)}{2}} \int_{\mathcal{M}_n^{\mathbb{C}}} \sqrt{\sum_{i=1}^{n} \prod_{\substack{j=1\\ j\neq i}}^{n} h\prime(\mu_j)} \left(\prod_{1\leqslant i<j\leqslant n} M(\mu_i(D), \mu_j(D)) \right)^2 \mathrm{d}\lambda_{\dim(\mathcal{M}_n^{\mathbb{C}})}(D).$$

4. Volume of the state space of qubits

In the space of qubits we choose the Stokes parametrization, i.e. we write a state $D \in \mathcal{M}_2$ in the form

$$D = \frac{1}{2}\begin{pmatrix} 1+x & y+\mathrm{i}\,z \\ y+\mathrm{i}\,z & 1-x \end{pmatrix}.$$

Using these coordinates, the spaces $\mathcal{M}_2^{\mathbb{C}}$ and $\mathcal{M}_2^{\mathbb{R}}$ can be identified with the unit ball in the Euclidean spaces $\mathbb{R}^3$ and $\mathbb{R}^2$. The metric g_f which is generated by an operator monotone function in this coordinate system is

$$g_f(x,y,z) = \begin{pmatrix} \frac{1}{4\lambda_1\lambda_2} & 0 & 0 \\ 0 & \frac{m(\lambda_1,\lambda_2)}{2} & 0 \\ 0 & 0 & \frac{m(\lambda_1,\lambda_2)}{2} \end{pmatrix} \qquad g_f(x,y) = \begin{pmatrix} \frac{1}{4\lambda_1\lambda_2} & 0 \\ 0 & \frac{m(\lambda_1,\lambda_2)}{2} \end{pmatrix}.$$

The volume is an integral on the unit ball, which is in spherical and polar coordinates

$$V(\mathcal{M}_2^{\mathbb{C}}) = 4\pi \int_0^1 \frac{r^2}{\sqrt{1-r^2}(1+r) f\left(\frac{1-r}{1+r}\right)} \mathrm{d}r,$$

$$V(\mathcal{M}_2^{\mathbb{R}}) = 2\pi \int_0^1 \frac{r}{\sqrt{1-r}(1+r)\sqrt{f\left(\frac{1-r}{1+r}\right)}} \mathrm{d}r,$$

respectively.

Corollary 1. *The volume of the space* $(\mathcal{M}_2, g_f)$ *where the metric* g_f *is generated by an operator monotone function* f *is*

$$V(\mathcal{M}_2^{\mathbb{C}}) = 2\pi \int_0^1 \left(\frac{1-t}{1+t}\right)^2 \frac{1}{\sqrt{t}\, f(t)}\, \mathrm{d}t$$

$$V(\mathcal{M}_2^{\mathbb{R}}) = \sqrt{2}\pi \int_0^1 \frac{1-t}{1+t} \frac{1}{\sqrt{t+t^2}\sqrt{f(t)}}\, \mathrm{d}t.$$

Here are some operator monotone functions which generate monotone metrics from [3, 16, 17] and the corresponding volumes:

$f(x)$:	$V\left(\mathcal{M}_2^{\mathbb{C}}\right)$:	$V\left(\mathcal{M}_2^{\mathbb{R}}\right)$:
$\frac{1+x}{2}$	π^2	2π
$\frac{2x}{1+x}$	∞	∞
$\frac{x-1}{\log x}$	$2\pi^2$	~ 8.298
$\sqrt{x}$	∞	4π
$\frac{1}{4}(\sqrt{x}+1)^2$	$4\pi(\pi-2)$	$4\pi(2-\sqrt{2})$
$\frac{2\sqrt{x}(x-1)}{(1+x)\log x}$	∞	~ 19.986
$\frac{2(x-1)^2}{(1+x)(\log x)^2}$	$\frac{\pi^4}{2}$	~ 11.51
$\frac{x}{2}\left(\frac{1}{\alpha x+1-\alpha}+\frac{1}{(1-\alpha)x+\alpha}\right)$	∞	∞
$\frac{2}{x+1}(\beta x+1-\beta)((1-\beta)x+\beta)$	$\pi^2\frac{1-2\sqrt{\beta-\beta^2}}{(1-2\beta)^2\sqrt{\beta-\beta^2}}$	$? < \infty$
$\frac{2x^{\gamma+\frac{1}{2}}}{1+x^{2\gamma}}$	∞	$? < \infty.$

The parameters lie in the interval $\alpha \in \left]0, \frac{1}{2}\right]$, $\beta \in \left]0, \frac{1}{2}\right[$ and $\gamma \in \left[0, \frac{1}{2}\right]$. We have some open questions about this list. For every function in this list the complex state space has a greater volume; a natural question is: is this necessary? It seems that if for a function f the volume $V\left(\mathcal{M}_2^{\mathbb{C}}\right)$ is finite, then for the transpose function $f^{\perp}(x) = \frac{x}{f(x)}$ the volume is infinity, except for the function $f(x) = \sqrt{x}$; in this case $f = f^{\perp}$. Is this true in general? For some functions, the difference between the volumes is infinity. What can be the statistical meaning of this phenomenon?

The origin of the infinite volume of the space $\mathcal{M}_2^{\mathbb{C}}$ can be understood partially by the help of a representation theorem for operator monotone functions. This representation theorem is due to Löwner [12], but we use a modified version from [8].

Theorem 8. *The map* $\mu \mapsto f$*, defined by*

$$f(x) = \int_0^1 \frac{x}{(1-t)x+t}\,\mathrm{d}\mu(t), \qquad \text{for} \quad x > 0,$$

establishes a bijection between the class of positive Radon measures on $[0, 1]$ *and the class of operator monotone functions. The function* f *fulfills the condition* $f(x) = xf(x^{-1})$ *for every positive* x *if and only if for every* $s \in [0, 1]$ *the equality* $\mu([0, s]) = \mu([1-s, 1])$ *holds.*

If f is an operator monotone function, then its transpose $f^{\perp}$ is monotone too [5]. That is, $1/f(x)$ can also be written in the form

$$\int_0^1 \frac{1}{(1-t)x+t}\,\mathrm{d}\mu(t),$$

where μ is a probability measure on $[0,1]$ with the symmetric property $\mu([0,s]) = \mu([1-s,1])$. Substituting this representation of f into the volume formula for $\mathcal{M}_2^{\mathbb{C}}$, we have that if μ is the corresponding symmetric measure for the function $f^{\perp}$, then the volume of the manifold $\left(\mathcal{M}_2^{\mathbb{C}}, g_f\right)$ is

$$V = \int_0^1 \frac{2}{2z-1} - \frac{\pi}{(2z-1)^2} + \frac{\arccos(2z-1)}{(2z-1)^2\sqrt{z-z^2}}\,\mathrm{d}\mu(z).$$

The integrand is continuous, monotonically decreasing and has a series expansion

$$\pi\frac{1}{\sqrt{z}} - (4+\pi) + \frac{9\pi}{2}\sqrt{z} - 4\left(\frac{10}{3}+\pi\right)z + \cdots$$

near the origin. Its integral with respect to a symmetric probability measure is infinity if and only if

$$\int_0^1 \frac{1}{\sqrt{z}}\,\mathrm{d}\mu(z) = \infty$$

holds. So the volume of the complex state space of qubits is infinity if the metric is generated by a symmetric probability measure which is concentrated in some sense at the ends of the interval [0, 1].

If we consider the space of qubits with a pull-back metric, then we have the following corollary by using the above-mentioned techniques.

Corollary 2. *For an admissible function f, let us consider the real and complex space $\mathcal{M}_2$ with the pull-back metric. The volume of this space is the following:*

$$V\left(\mathcal{M}_2^{\mathbb{R}}\right) = \frac{\pi}{\sqrt{2}}\int_0^1 \sqrt{f'\left(\frac{1+r}{2}\right)^2 + f'\left(\frac{1-r}{2}\right)^2}\left(f\left(\frac{1+r}{2}\right) - f\left(\frac{1-r}{2}\right)\right)\mathrm{d}r$$

$$V\left(\mathcal{M}_2^{\mathbb{C}}\right) = \pi\int_0^1 \sqrt{f'\left(\frac{1+r}{2}\right)^2 + f'\left(\frac{1-r}{2}\right)^2}\left(f\left(\frac{1+r}{2}\right) - f\left(\frac{1-r}{2}\right)\right)^2\mathrm{d}r.$$

Acknowledgments

This work was supported by Hungarian Scientific Research Fund (OTKA) contract T046599, TS049835, and EU Network ‘QP-Applications’ contract number HPRN–CT–2002–00729.

Appendix A. Volume of the state space

Theorem 1. *For every $k \in \mathbb{N}$, the volume of the state spaces $\mathcal{M}_{2k}^{\mathbb{R}}$ and $\mathcal{M}_{2k+1}^{\mathbb{R}}$ are*

$$V\left(\mathcal{M}_{2k}^{\mathbb{R}}\right) = \frac{\pi^{k^2}}{2^{k^2+k}}\frac{(2k)!}{k!(2k^2+k-1)!}\prod_{i=1}^{k-1}(2i)!$$

$$V\left(\mathcal{M}_{2k+1}^{\mathbb{R}}\right) = \left(\frac{\pi}{2}\right)^{k^2+k}\frac{(2k)!}{(k-1)!(2k^2+3k)!}\prod_{i=1}^{k-1}(2i)!.$$

Proof. A self-adjoint $n \times n$ matrix with real entries A is in $\mathcal{M}_n^{\mathbb{R}}$ if and only if

$$\forall i \in \{1,\ldots,n\}: \quad \det(A_i) > 0, \quad \sum_{k=1}^{n} a_k = 1,$$

where $(a_i)_{i=1,\ldots,n}$ are the diagonal elements of A. First we assume that the matrix of the diagonal elements D is given. If A_{n-1} is fixed, then

$$V(A_{n-1}) = \int_{\mathrm{E}^{\mathbb{R}}_{n-1}(T_{n-1},a_n\det(A_{n-1}))} 1\,\mathrm{d}\lambda_{n-1} = a_n^{(n-1)/2} F_{n-2} G_{n-2,0} \sqrt{\det(A_{n-1})}.$$

If A_{n-2} is fixed, then

$$\begin{aligned} V(A_{n-2}) &= \int_{\mathrm{E}^{\mathbb{R}}_{n-2}(T_{n-2},a_{n-1}\det(A_{n-2}))} V(A_{n-1})\,\mathrm{d}\lambda_{n-2} \\ &= a_n^{\frac{n-1}{2}} F_{n-2} G_{n-2,0} \int_{\mathrm{E}^{\mathbb{R}}_{n-2}(T_{n-2},a_{n-1}\det(A_{n-2}))} (a_{n-1}\det(A_{n-2}) - \langle x, T_{n-2}x\rangle)^{\frac{1}{2}}\,\mathrm{d}\lambda_{n-2}(x) \\ &= a_{n-1}^{\frac{n-1}{2}} a_n^{\frac{n-1}{2}} F_{n-2} F_{n-3} G_{n-2,0} G_{n-3,1/2} \det(A_{n-2}). \end{aligned}$$

In general if A_{n-k} is fixed, then

$$V(A_{n-k}) = \prod_{i=1}^{k} \left(a_{n+1-i}^{(n-1)/2} F_{n-1-i} G_{n-1-i,(i-1)/2}\right) \det(A_{n-k})^{\frac{k}{2}},$$

because this equation is correct for $k = 1$ and by induction

$$\begin{aligned} \int_{\mathrm{E}^{\mathbb{R}}_{n-k-1}(T_{n-k-1},a_{n-k}\det(A_{n-k-1}))} V(A_{n-k})\,\mathrm{d}\lambda_{n-k-1} &= \prod_{i=1}^{k} \left(a_{n+1-i}^{(n-1)/2} F_{n-1-i} G_{n-1-i,(i-1)/2}\right) \\ &\quad\times \int_{\mathrm{E}^{\mathbb{R}}_{n-k-1}(T_{n-k-1},a_{n-k}\det(A_{n-k-1}))} (a_{n-k}\det(A_{n-k-1}) - \langle x, T_{n-k-1}x\rangle)^{\frac{k}{2}}\,\mathrm{d}\lambda_{n-k-1}(x) \\ &= \prod_{i=1}^{k} \left(a_{n+1-i}^{(n-1)/2} F_{n-1-i} G_{n-1-i,(i-1)/2}\right) \\ &\quad\times a_{n-k}^{(n-1)/2} F_{n-k-2} G_{n-k-2,k/2} \det(A_{n-k-1})^{\frac{k+1}{2}} = V(A_{n-k-1}). \end{aligned}$$

It means that

$$V(A_1) = \left(\prod_{i=0}^{n-2} F_i\right) \left(\prod_{i=1}^{n-1} G_{n-1-i,(i-1)/2}\right) \left(\prod_{i=1}^{n} a_n\right)^{(n-1)/2}.$$

So the volume of the real state space is

$$V\left(\mathcal{M}_n^{\mathbb{R}}\right) = \left(\prod_{i=1}^{n-1} F_{i-1} G_{n-1-i,(i-1)/2}\right) \int_{\Delta_{n-1}} \left(\prod_{i=1}^{n} a_n\right)^{(n-1)/2} \mathrm{d}\lambda_{n-1}(a).$$

The integral in this equation is

$$\frac{\Gamma\left(\frac{n+1}{2}\right)^n}{\Gamma\left(\frac{n^2+n}{2}\right)}$$

and the product is

$$\varphi = \left(\prod_{i=1}^{n-1} F_{i-1} G_{n-1-i,(i-1)/2}\right) = \frac{\pi^{\frac{n^2-n}{4}}}{2^{n-1}} \frac{(n-1)!}{\Gamma\left(\frac{n+1}{2}\right)^{n-1}} \prod_{i=1}^{n-1} \frac{\Gamma\left(\frac{i+1}{2}\right)\Gamma\left(\frac{n-i}{2}\right)}{\Gamma\left(\frac{i}{2}+1\right)}.$$

If $n = 2k + 1$, then

$$\varphi = \frac{\pi^{k^2+\frac{k}{2}}}{2^{2k}} \frac{(2k)!}{(k!)^{2k+1}} \prod_{i=1}^{2k} \Gamma\left(\frac{i}{2}\right)$$

which can be simplified using the equality $\Gamma(i)\Gamma(i+1/2) = \dfrac{\sqrt{\pi}(2i)!}{2^{2i}i}$ to

$$\varphi = \left(\frac{\pi}{2}\right)^{k^2+k} \frac{k(2k)!}{(k!)^{2k+2}} \prod_{i=1}^{k-1} (2i)!.$$

If $n = 2k$, then using the same identity for the function Γ, we have

$$\varphi = \pi^{k^2-k} 2^{3k^2-k} \left(\frac{k!}{(2k)!}\right)^{2k-1} \prod_{i=1}^{k-1} (2i)!.$$

□

Theorem 2. *For every $n \in \mathbb{N}$, the volume of the state space $\mathcal{M}_n^{\mathbb{C}}$ is*

$$V\big(\mathcal{M}_n^{\mathbb{C}}\big) = \frac{\pi^{\frac{n(n-1)}{2}}}{(n^2-1)!} \prod_{i=1}^{n-1} i!.$$

Proof. The proof is similar to the real case, except that we have to take into account that the dimension of the parameter space of a matrix element A_{ij} for $i \neq j$ indices is 2. If A_{n-1} is fixed, then

$$V(A_{n-1}) = \int_{\mathrm{E}_{2n-2}^{\mathbb{C}}(T_{n-1}, a_n \det(A_{n-1}))} 1 \,\mathrm{d}\lambda_{2n-2} = a_n^{n-1} F_{2n-3} G_{2n-3,0} \det(A_{n-1})$$

and in general if A_{n-k} is fixed, then

$$V(A_{n-k}) = \prod_{i=1}^{k} \big(a_{n+1-i}^{n-1} F_{2n-1-2i} G_{2n-1-2i,i-1}\big) \det(A_{n-k})^k.$$

The volume of the complex state space is

$$V\big(\mathcal{M}_n^{\mathbb{C}}\big) = \left(\prod_{i=1}^{n-1} F_{2i-1} G_{2n-1-2i,i-1}\right) \int_{\Delta_{n-1}} \left(\prod_{i=1}^{n} a_n\right)^{n-1} \mathrm{d}\lambda_{n-1}(a),$$

where the product is

$$\frac{\pi^{\frac{n^2-n}{2}}}{((n-1)!)^n} \prod_{i=1}^{n-1} i!$$

and the integral is

$$\frac{((n-1)!)^n}{(n^2-1)!}.$$

□

Theorem 3. *For every $n \in \mathbb{N}$, the volume of the state space $\mathcal{M}_n^{\mathbb{H}}$ is*

$$V\big(\mathcal{M}_n^{\mathbb{H}}\big) = \frac{(2n-2)!\pi^{n^2-n}}{(2n^2-n-1)!} \prod_{i=1}^{n-2} (2i)!.$$

Proof. If A_{n-1} is fixed, then

$$V(A_{n-1}) = a_n^{2n-2} F_{4n-5} G_{4n-5,0} \det(A_{n-1})^2$$

and in general if A_{n-k} is fixed, then

$$V(A_{n-k}) = \prod_{i=1}^{k} \big(a_{n+1-i}^{2n-2} F_{4n-1-4i} G_{4n-1-4i,2i-2}\big) \det(A_{n-k})^{2k}.$$

The volume of the quaternionic state space is

$$V\left(\mathcal{M}_n^{\mathbb{H}}\right) = \left(\prod_{i=1}^{n-1} F_{4i-1} G_{4n-1-4i,2i-2}\right) \int_{\Delta_{n-1}} \left(\prod_{i=1}^{n} a_n\right)^{2n-2} \mathrm{d}\lambda_{n-1}(a),$$

where the product is

$$\frac{\pi^{n^2-n}}{((2n-2)!)^{n-1}} \prod_{i=1}^{n-1}(2i-2)!$$

and the integral is

$$\frac{((2n-2)!)^n}{(2n^2-n-1)!}.$$

□

Theorem 4. *For every parameter* $\alpha \in \mathbb{R}^+$ *and* $n \in \mathbb{N}$ *the expected value of the function* $\det^\alpha$ *on the state spaces* $\mathcal{M}_n^{\mathbb{R}}, \mathcal{M}_n^{\mathbb{C}}$ *and* $\mathcal{M}_n^{\mathbb{H}}$ *with respect to the normalized Lebesgue measures* $\mu_{\mathbb{R}}, \mu_{\mathbb{C}}$ *and* $\mu_{\mathbb{H}}$ *are*

$$\int_{\mathcal{M}_n^{\mathbb{R}}} \det(A)^\alpha \, \mathrm{d}\mu_{\mathbb{R}}(A) = \frac{\Gamma\left(\frac{n^2+n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)} \frac{\Gamma\left(\frac{n+1}{2}+\alpha\right)}{\Gamma\left(\frac{n^2+n}{2}+n\alpha\right)} \prod_{i=1}^{n-1} \frac{\Gamma\left(\frac{i+1}{2}+\alpha\right)}{\Gamma\left(\frac{i+1}{2}\right)}$$

$$\int_{\mathcal{M}_n^{\mathbb{C}}} \det(A)^\alpha \, \mathrm{d}\mu_{\mathbb{C}}(A) = \frac{(n^2-1)!}{(n-1)!} \frac{\Gamma(n+\alpha)}{\Gamma(n^2+n\alpha)} \prod_{i=1}^{n-1} \frac{\Gamma(i+\alpha)}{\Gamma(i)}$$

$$\int_{\mathcal{M}_n^{\mathbb{H}}} \det(A)^\alpha \, \mathrm{d}\mu_{\mathbb{H}}(A) = \frac{\Gamma(2n^2-n)}{\Gamma(2n-1)} \frac{\Gamma(2n+\alpha-1)}{\Gamma(2n^2-n+\alpha n)} \prod_{i=1}^{n-1} \frac{\Gamma(2i-1+\alpha)}{\Gamma(2i-1)}.$$

Proof. The proofs are similar, so we just prove the theorem for the real case only. First we compute the integral with respect to the Lebesgue measure, and we divide the result with the volume of the state space. The method is the same as in the previous theorems, so if A_{n-1} is given, then

$$\begin{aligned} V(A_{n-1}) &= \int_{\mathrm{E}_{n-1}^{\mathbb{R}}(T_{n-1}, a_n \det(A_{n-1}))} \det(A_n)^\alpha \, \mathrm{d}\lambda_{n-1} \\ &= \int_{\mathrm{E}_{n-1}^{\mathbb{R}}(T_{n-1}, a_n \det(A_{n-1}))} (a_n \det(A_{n-1}) - \langle x, T_{n-1}x\rangle)^\alpha \, \mathrm{d}\lambda_{n-1}(x) \\ &= a_n^{(n-1)/2+\alpha} F_{n-2} G_{n-2,\alpha} \det(A_{n-1})^{\frac{1}{2}+\alpha}, \end{aligned}$$

and the general formula is

$$V(A_{n-k}) = \prod_{i=1}^{k} \left(a_{n+1-i}^{(n-1)/2+\alpha} F_{n-1-i} G_{n-1-i,(i-1)/2+\alpha}\right) \det(A_{n-k})^{\frac{k}{2}+\alpha}.$$

The integral of the function $\det^\alpha$ with respect to the Lebesgue measure is

$$\int_{\mathcal{M}_n^{\mathbb{R}}} \det(A)^\alpha \, \mathrm{d}\lambda_{\dim(\mathcal{M}_n^{\mathbb{R}})}(A) = \left(\prod_{i=1}^{n-1} F_{i-1} G_{n-1-i,(i-1)/2+\alpha}\right) \int_{\Delta_{n-1}} \left(\prod_{i=1}^{n} a_n\right)^{\frac{n-1}{2}+\alpha} \mathrm{d}\lambda_{n-1}(a).$$

Dividing it with $V\left(\mathcal{M}_n^{\mathbb{R}}\right)$ and after some simplification, we get the formula which is in the theorem. □

References

[1] Amari S 1985 *Differential-Geometrical Methods in Statistics (Lecture Notes in Statistics* vol 28*)* (Berlin: Springer)

[2] Amari S and Nagaoka H 2000 Methods of information geometry *Translations of Mathematical Monographs* vol 191 (Providence, RI: American Mathematical Society)

[3] Andai A 2003 Monotone Riemannian metrics on density matrices with nonmonotone scalar curvature *J. Math. Phys.* **44** 3675–88

[4] Andai A 2006 On the curvature of the quantum state space with pull-back metrics *Preprint* math-ph/0604031

[5] Bhatia R 1997 Matrix analysis *Graduate Texts in Mathematics* vol 169 (New York: Springer)

[6] Čencov N N 1982 Statistical decision rules and optimal inference *Translations of Mathematical Monographs* vol 53 (Providence, RI: American Mathematical Society)

[7] Gibilisco P and Isola T 2005 On the monotonicity of scalar curvature in classical and quantum information geometry *J. Math. Phys.* **46** 023501

[8] Gibilisco P and Isola T 2001 Monotone metrics on statistica manifolds of density matrices by geometry of non-commutative l^2-space *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **1** 129–40

[9] Gibilisco P and Pistone G 1999 Analytical and geometrical properties of statistical connections in information geometry *Mathematical Theory of Networks and Systems* ed A Beghi *et al* (Padova: Il Poigrafo) pp 881–914

[10] Hall M J W 1998 Random quantum correlations and density operator distributions *Phys. Lett.* A **242** 123–9

[11] Hiai F, Petz D and Toth G 1996 Curvature in the geometry of canonical correlation *Stud. Sci. Math. Hung.* **32** 235–49

[12] Löwner K 1934 Über monotone Matrixfunktionen *Math. Z.* **38** 177–216

[13] Mehta M L 1991 *Random Matrices* 2nd edn (Boston, MA: Academic)

[14] Michor P W, Petz D and Andai A 2000 On the curvature of a certain Riemannian space of matrices *Quantum Probab. Infin. Dimens. Anal.* **3** 199–212

[15] Morozova E A and Čencov N N 1989 Markov invariant geometry on state manifolds *Current Problems in Mathematics Newest Results (Russian)* vol 36 (Moscow: Akad Nauk SSSR. Vsesoyuz. Inst. Nauchn. i Tekhn. Inform.) pp 69–102

[16] Petz D 1996 Monotone metrics on matrix spaces *Linear Algebr. Appl.* **244** 81–96

[17] Petz D 1994 Geometry of canonical correlation on the state space of a quantum system *J. Math. Phys.* **35** 780–95

[18] Rao C R 1945 Information and accuracy attainable in the estimation of statistical parameters *Rev. Bull. Calcutta Math. Soc.* **37** 89–91

[19] Slater P B 1999 Hall normalization constants for the Bures volumes of the n-state quantum systems *J. Phys. A: Math. Gen.* **32** 8231–46

[20] Slater P B 1999 A priori probabilities of separable quantum states *J. Phys. A: Math. Gen.* **32** 5261–75

[21] Slater P B 1998 Comparative noninformativities of quantum priors based on monotone metrics *Phys. Lett.* A **247** 1–8

[22] Slater P B 1996 The quantum Jeffrey's prior/Bures metric volume element for squeezed thermal states and a universal coding conjecture *J. Phys. A: Math. Gen.* **29** L601–5

[23] Slater P B 1996 Quantum Fisher–Bures information of two-level systems and a three-level extension *J. Phys. A: Math. Gen.* **29** L271–5

[24] Slater P B 1996 Applications of quantum and classical Fisher information to two-level complex and quaternionic and three-level complex systems *J. Math. Phys.* **37** 2682–93

[25] Slater P B 2005 Silver mean conjectures for 15-d volumes and 14-d hyperareas of the separable two-qubit systems *J. Geom. Phys.* **53** 74–97

[26] Sommers H and Zyczkowski K 2003 Bures volume of the set of mixed quantum states *J. Phys. A: Math. Gen.* **36** 10083–100

[27] Streater R F 2001 Classical and quantum info-manifolds *Sūrikaisekikenkyūsho Kōkyūroku* **1196** 32–51

[28] Zyczkowski K and Sommers H 2003 Hilbert–Schmidt volume of the set of mixed quantum states *J. Phys. A: Math. Gen.* **36** 10115–30